

ESTIMATES OF EIGENVALUES OF LAPLACIAN BY A REDUCED NUMBER OF SUBSETS

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ABSTRACT. Chung-Grigor'yan-Yau's inequality describes upper bounds of eigenvalues of Laplacian in terms of subsets ("input") and their volumes. In this paper we will show that we can reduce ginpuh in Chung-Grigor'yan-Yau's inequality in the setting of Alexandrov spaces satisfying $CD(0, \infty)$. We will also discuss a related conjecture for some universal inequality among eigenvalues of Laplacian.

1. INTRODUCTION

The study of eigenvalues of Laplacian is now a classical but important subject in mathematics. It is closely related with geometry of underlying spaces such as curvature, volume, diameter, closed geodesics, and etc, see [Cha84]. In this paper we prove the following. For two subsets A, B in a metric space (X, d) we denote

$$d(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}.$$

Theorem 1.1. *There exists a universal and numerical constant $c > 0$ satisfying the following property. Let (X, μ) be an weighted compact finite-dimensional Alexandrov space satisfying $CD(0, \infty)$ and $\mu(X) = 1$. For any $l+1$ Borel subsets A_0, A_1, \dots, A_l with $l \leq k$, the k th eigenvalue $\lambda_k(X, \mu)$ of the weighted Laplacian has the estimate*

$$(1.1) \quad \lambda_k(X, \mu) \leq \frac{c^{k-l+1}}{\min_{i \neq j} d(A_i, A_j)^2} \left(\max_{i \neq j} \log \frac{1}{\mu(A_i)\mu(A_j)} \right)^2.$$

Here an *Alexandrov space* is a complete geodesic metric space with local 'sectional curvature' bounds introduced by A. D. Alexandrov in terms of comparison properties of geodesic triangles. The condition $CD(0, \infty)$ stands for the space (X, μ) has nonnegative 'Ricci curvature' (see Section 3).

This inequality was first proved by Gromov and V. Milman in the case where $k = l = 1$ without curvature assumption ([GM83]). It is equivalent to an exponential concentration inequality (see Lemma 4.3). Chung, Grigor'yan, and Yau generalized their result to the case where $k = l$ ([CGY96, CGY97]). Although Chung-Grigor'yan-Yau's setting was for manifolds, their proof also works for Alexandrov spaces without any changes. See Section 2.4 for details. The crucial point of the above theorem is that one can reduce the number of subsets ("input") in a dimension-free way under assuming $CD(0, \infty)$.

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Remark 1.2. (1) The purpose in the previous paper [Fun13] was to use Theorem 1.1 to understand the relationships between the eigenvalues $\lambda_k(M, \mu)$ of the weighted Laplacian for different k , where (M, μ) is a compact weighted Riemannian manifold (M, μ) having nonnegative Bakry-Émery Ricci curvature. Precisely, the author obtained the universal inequalities $\lambda_k(M, \mu) \leq c^k \lambda_1(M, \mu)$ among the eigenvalues. After the paper [Fun13] was written Liu proved the sharp universal inequality $\lambda_k(X, \mu) \leq ck^2 \lambda_1(X, \mu)$ for weighted compact finite-dimensional Alexandrov space (X, μ) satisfying $\text{CD}(0, \infty)$ ([Liu]). He pointed out that the so called the improved Cheeger inequality holds via the same proof for graph setting in [KLLGT13] and combining with the Buser-Ledoux inequality implies the above sharp inequality.

(2) In the previous paper [Fun13] the author proved Theorem 1.1 only for compact weighted manifolds. The proof in [Fun13] implicitly uses the smooth structure of the underlying spaces (see Section 3 for details). Theorem 1.1 in the present paper avoids the issue and is stronger than the one in [Fun13]. In fact, since our Alexandrov spaces are metric spaces with “local” sectional curvature bounds, our setting includes any compact weighted Riemannian manifolds satisfying $\text{CD}(0, \infty)$. The author will not publish the paper [Fun13] from any journal.

(3) One can extend Theorem 1.1 for spaces with lower negative Ricci curvature bounds K if we add some restriction in diameter according to the lower bound K and if we allow that the constant c depends on diameter and K . See the proof of Theorem 1.1. In Theorem 1.1 the lower bounds of Ricci curvature is necessary as was remarked in [FS13]. Some ‘dumbbell space’ gives a counterexample, see [FS13, Example 4.9] for details.

(4) Theorem 1.1 is true also in the case where X has non-empty boundary. In that case we implicitly assume the Neumann boundary condition.

In the proof of Theorem 1.1 we will work on the notion of ‘*separation*’, which is regarded as a generalization of the concentration of measure phenomenon (see Subsection 2.2). It tells the information whether or not there exists a pair which are not separated in some sense among any $k+1$ -tuple subsets with a fixed volume. The idea of the proof of Theorem 1.1 will be discussed in Section 3 in details.

2. PRELIMINARIES

We review some basics needed in this paper.

2.1. Lévy radius. Let X be an *mm-space*, i.e., a complete separable metric space with a Borel probability measure μ_X .

Let $f : X \rightarrow \mathbb{R}$ a Borel measurable function. A real number m_f is called a *median* of f if it satisfies that

$$\mu_X(\{x \in X \mid f(x) \geq m_f\}) \geq 1/2 \quad \text{and} \quad \mu_X(\{x \in X \mid f(x) \leq m_f\}) \geq 1/2.$$

The set of all median of the function f is a bounded closed interval $[a_f, b_f]$. We define $lm(f; \mu_X) := (a_f + b_f)/2$.

Definition 2.1 (Lévy radius). For $\kappa > 0$, we define the *Lévy radius* $\text{LeRad}(X; -\kappa)$ of an mm-space X as the infimum of $\rho > 0$ such that every 1-Lipschitz function $f : X \rightarrow \mathbb{R}$ satisfies that

$$\mu_X(\{x \in X \mid |f(x) - \text{lm}(f; \mu_X)| \geq \rho\}) \leq \kappa.$$

Refer to [Gro99], [Led01] for the background of Lévy radius.

2.2. Separation distance. We define the separation distance which plays an important role in the proof of Theorem 1.1. The separation distance was introduced by Gromov in [Gro99].

Definition 2.2 (Separation distance). For any $\kappa_0, \kappa_1, \dots, \kappa_k \geq 0$ with $k \geq 1$, we define the (k) -*separation distance* $\text{Sep}(X; \kappa_0, \kappa_1, \dots, \kappa_k)$ of X as the supremum of $\min_{i \neq j} d(A_i, A_j)$, where A_0, A_1, \dots, A_k are any Borel subsets of X satisfying that $\mu_X(A_i) \geq \kappa_i$ for all $i = 0, 1, \dots, k$.

It is immediate from the definition that if $\kappa_i \geq \tilde{\kappa}_i$ for each $i = 0, 1, \dots, k$, then

$$\text{Sep}(X; \kappa_0, \kappa_1, \dots, \kappa_k) \leq \text{Sep}(X; \tilde{\kappa}_0, \tilde{\kappa}_1, \dots, \tilde{\kappa}_k).$$

Note that if the support of μ_X is connected, then

$$\text{Sep}(X; \kappa_0, \kappa_1, \dots, \kappa_k) = 0$$

for any $\kappa_0, \kappa_1, \dots, \kappa_k > 0$ such that $\sum_{i=0}^k \kappa_i > 1$.

We denote the closed r -neighborhood of a subset A in a metric space by $C_r(A)$.

Lemma 2.3. *Let X be an mm-space and $k \geq 1$. Put $r := \text{Sep}(X, \kappa_0, \kappa_1, \dots, \kappa_k)$. Assume that k Borel subsets A_0, A_1, \dots, A_{k-1} of X satisfy $\mu_X(A_i) \geq \kappa_i$ for every $i = 0, 1, \dots, k-1$ and $d(A_i, A_j) > r$ for every $i \neq j$. Then we have*

$$\mu_X\left(\bigcup_{i=0}^{k-1} C_r(A_i)\right) \geq 1 - \kappa_k.$$

Proof. Suppose that for some $\varepsilon_0 > 0$,

$$\mu_X\left(\bigcup_{i=0}^{k-1} C_{r+\varepsilon_0}(A_i)\right) \leq 1 - \kappa_k.$$

Putting $A_k := X \setminus \bigcup_{i=0}^{k-1} C_{r+\varepsilon_0}(A_i)$ we have $\mu_X(A_k) \geq \kappa_k$ and $d(A_k, A_i) \geq r + \varepsilon_0$ for any $i = 0, 1, \dots, k-1$. Thus we get

$$r < \min_{i \neq j} d(A_i, A_j) \leq \text{Sep}(X; \kappa_0, \kappa_1, \dots, \kappa_k) = r,$$

which is a contradiction. Hence $\mu_X(\bigcup_{i=0}^{k-1} C_{r+\varepsilon}(A_i)) > 1 - \kappa_k$ for any $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$ we obtain the conclusion. \square

Lemma 2.4. *For any $\kappa > 0$ we have*

$$\text{LeRad}(X; -\kappa) \leq \text{Sep}(X; \kappa/2, \kappa/2)$$

Proof. The lemma follows from a result in [Gro99] (cf. [FS13, Lemmas 2.22, 2.24]). \square

2.3. Three distances between probability measures. Let X be a complete separable metric space. We denote by $\mathcal{P}(X)$ the set of Borel probability measures on X .

Definition 2.5 (Prohorov distance). Given two measures $\mu, \nu \in \mathcal{P}(X)$ and $\lambda \geq 0$, we define the *Prohorov distance* $\text{di}_\lambda(\mu, \nu)$ as the infimum of $\varepsilon > 0$ such that

$$(2.1) \quad \mu(C_\varepsilon(A)) \geq \nu(A) - \lambda\varepsilon \text{ and } \nu(C_\varepsilon(A)) \geq \mu(A) - \lambda\varepsilon$$

for any Borel subsets $A \subseteq X$.

For any $\lambda \geq 0$, the function di_λ is a complete separable distance function on $\mathcal{P}(X)$. If $\lambda > 0$, then the topology on $\mathcal{P}(X)$ determined by the Prohorov distance function di_λ coincides with that of the weak convergence (see [Bil99, Section 6]). The distance functions di_λ for all $\lambda > 0$ are equivalent to each other. Also it is known that if $\mu(C_\varepsilon(A)) \geq \nu(A) - \lambda\varepsilon$ for any Borel subsets A of X , then $\text{di}_\lambda(\mu, \nu) \leq \varepsilon$. In other words, the second inequality in (2.1) follows from the first one (see [Bil99, Section 6]).

For $(x, y) \in X \times X$, we put $\text{proj}_1(x, y) := x$ and $\text{proj}_2(x, y) := y$. For two finite Borel measures μ and ν on X , we write $\mu \leq \nu$ if $\mu(A) \leq \nu(A)$ for any Borel subset $A \subseteq X$. A finite Borel measure π on $X \times X$ is called a *partial transportation* from $\mu \in \mathcal{P}(X)$ to $\nu \in \mathcal{P}(X)$ if $(\text{proj}_1)_*(\pi) \leq \mu$ and $(\text{proj}_2)_*(\pi) \leq \nu$. Note that we do not assume π to be a probability measure. For a partial transportation π from μ to ν , we define its *deficiency* $\text{def } \pi$ by $\text{def } \pi := 1 - \pi(X \times X)$. Given $\varepsilon > 0$, the partial transportation π is called an ε -*transportation* from μ to ν if it is supported in the subset

$$\{(x, y) \in X \times X \mid d(x, y) \leq \varepsilon\}.$$

Definition 2.6 (Transportation distance). Let $\lambda \geq 0$. For two probability measures $\mu, \nu \in \mathcal{P}(X)$, we define the *transportation distance* $\text{Tra}_\lambda(\mu, \nu)$ between μ and ν as the infimum of $\varepsilon > 0$ such that there exists an ε -transportation π from μ to ν satisfying $\text{def } \pi \leq \lambda\varepsilon$.

The following theorem is due to V. Strassen.

Theorem 2.7 ([Vil03, Corollary 1.28], [Gro99, Section 3 $\frac{1}{2}$.10]). *For any $\lambda > 0$, we have*

$$\text{Tra}_\lambda = \text{di}_\lambda.$$

Let (X, d) be a complete metric space. We indicate by $\mathcal{P}^2(X)$ the set of all Borel probability measures $\nu \in \mathcal{P}(X)$ such that

$$\int_X d(x, y)^2 d\nu(y) < +\infty$$

for some $x \in X$.

Definition 2.8 ((L^2) -Wasserstein distance). For two probability measures $\mu, \nu \in \mathcal{P}^2(X)$, we define the L^2 -*Wasserstein distance* $d_2^W(\mu, \nu)$ between μ and ν as the infimum of

$$\left(\int_{X \times X} d(x, y)^2 d\pi(x, y) \right)^{1/2},$$

where $\pi \in \mathcal{P}^2(X \times X)$ runs over all *couplings* of μ and ν , i.e., probability measures π with the property that $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for any Borel subset $A \subseteq X$. It is known that this infimum is achieved by some transport plan, which we call an *optimal transport plan* for $d_2^W(\mu, \nu)$.

If the underlying space X is compact, then the topology on $\mathcal{P}(X)$ induced from the L^2 -Wasserstein distance function coincides with that of the weak convergence (see [Vil03, Theorem 7.12]).

2.4. Weighted Alexandrov spaces. We refer to [BBI01, BGP92] for basics of Alexandrov spaces and to [KS11, Section 4], [KMS01] for analysis of Alexandrov spaces.

Let X be a compact n -dimensional Alexandrov space and \mathcal{H}^n be its Hausdorff measure. Let μ be a probability measure on X defined by $d\mu := e^{-V} d\mathcal{H}^n$, where V is a function on X with a certain regularity condition (e.g., any Lipschitz continuous function is sufficient in the following argument). For the measure μ we define the *weighted Laplacian* (also called as the *Witten Laplacian*) Δ_μ by

$$\Delta_\mu := \Delta + \nabla V \cdot \nabla = -e^{-V} \operatorname{div}(e^{-V} \nabla \cdot),$$

where Δ is the nonnegative Laplacian. Δ_μ has discrete spectrum consisting of eigenvalues

$$0 = \lambda_0(X, \mu) < \lambda_1(X, \mu) \leq \dots \leq \lambda_k(X, \mu) \leq \dots$$

We remark that Chung-Grigor'yan-Yau's theorem (the case where $k = l$ in Theorem 1.1) holds for weighted compact finite-dimensional Alexandrov spaces. In fact, in the proof of the theorem we need only the Davies-Gaffney (weighted) heat kernel estimate

$$\int_A \int_B p_t(x, y) d\mu(x) d\mu(y) \leq \sqrt{\mu(A)\mu(B)} \exp\left(-\frac{d^2(A, B)}{4t}\right)$$

for any Borel subsets A, B and asymptotic expansion of (weighted) heat kernel by eigenvalues and eigenfunctions of Laplacian ([CGY96]). These are true for weighted compact finite-dimensional Alexandrov spaces ([Stu95], [KMS01]).

3. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1 we need to explain some useful tools from the theory of optimal transportation. Refer to [Vil03, Vil08] for more details.

Let (X, d) be a metric space. A rectifiable curve $\gamma : [0, 1] \rightarrow X$ is called a *geodesic* if its arclength coincides with the distance $d(\gamma(0), \gamma(1))$ and it has a constant speed, i.e., parameterized proportionally to the arclength. We say that a metric space is a *geodesic space* if any two points are joined by a geodesic between them. It is known that $(\mathcal{P}^2(X), d_2^W)$ is a compact geodesic space as soon as X is ([Stu06b, Proposition 2.10]).

Let X be a finite-dimensional Alexandrov space. For two probability measures $\mu_0, \mu_1 \in \mathcal{P}^2(M)$ which are absolutely continuous with respect to the Hausdorff measure, there is a unique geodesic $(\mu_t)_{t \in [0, 1]}$ between them with respect to the L^2 -Wasserstein distance function d_2^W ([McC01], [Ber08, Theorem 1.1]).

For an mm-space X let us denote by Γ the set of minimal geodesics $\gamma : [0, 1] \rightarrow X$ endowed with the distance

$$d_\Gamma(\gamma_1, \gamma_2) := \sup_{t \in [0, 1]} d(\gamma_1(t), \gamma_2(t)).$$

Define the *evaluation map* $e_t : \Gamma \rightarrow X$ for $t \in [0, 1]$ as $e_t(\gamma) := \gamma(t)$. A probability measure $\Pi \in \mathcal{P}(\Gamma)$ is called a *dynamical optimal transference plan* if the curve $\mu_t := (e_t)_*\Pi$, $t \in [0, 1]$, is a geodesic in $(\mathcal{P}^2(X), d_2^W)$. Then $\pi := (e_0 \times e_1)_*\Pi$ is an optimal coupling of μ_0 and μ_1 , where $e_0 \times e_1 : \Gamma \rightarrow X \times X$ is the “endpoints” map, i.e., $(e_0 \times e_1)(\gamma) := (e_0(\gamma), e_1(\gamma))$.

Lemma 3.1 ([LV09, Proposition 2.10]). *If (X, d) is locally compact, then any geodesic $(\mu_t)_{t \in [0, 1]}$ in $(\mathcal{P}^2(X), d_2^W)$ is associated with a dynamical optimal transference plan Π , i.e., $\mu_t = (e_t)_*\Pi$.*

Let μ and ν be two probability measures on a set X . We define the *relative entropy* $\text{Ent}_\mu(\nu)$ of ν with respect to μ as follows. If ν is absolutely continuous with respect to μ , writing $d\nu = \rho d\mu$, then

$$\text{Ent}_\mu(\nu) := \int_X \rho \log \rho d\mu,$$

otherwise $\text{Ent}_\mu(\nu) := \infty$.

Definition 3.2 (Curvature-dimension condition, [LV09], [Stu06a, Stu06b]). Let K be a real number. We say that a locally compact mm-space X satisfies the *curvature-dimension condition* $CD(K, \infty)$ if for any $\nu_0, \nu_1 \in \mathcal{P}^2(X)$ there exists a minimal geodesic $(\nu_t)_{t \in [0, 1]}$ in $(\mathcal{P}^2(X), d_2^W)$ from ν_0 to ν_1 such that

$$\text{Ent}_{\mu_X}(\nu_t) \leq (1 - t) \text{Ent}_{\mu_X}(\nu_0) + t \text{Ent}_{\mu_X}(\nu_1) - \frac{K}{2}(1 - t)t d_2^W(\nu_0, \nu_1)^2$$

for any $t \in [0, 1]$.

Example 3.3. (1) A complete weighted Riemannian manifold (M, μ) has Bakry-Émery Ricci curvature $\geq K$ for some $K \in \mathbb{R}$ if and only if (M, μ) satisfies $CD(K, \infty)$ ([CMS01, CMS06], [vRS05], [Stu05]).

(2) An n -dimensional Alexandrov space of curvature $\geq K$ satisfies $CD((n - 1)K, \infty)$ ([Pet11], [ZZ10]).

In the above definition, assume that both ν_0 and ν_1 are absolutely continuous with respect to μ_X . Then Jensen’s inequality applied to the convex function $r \mapsto r \log r$ gives

$$\begin{aligned} (3.1) \quad & \log \mu_X(\text{Supp } \nu_t) \\ & \geq -(1 - t) \int_X \rho_0 \log \rho_0 d\mu_X - t \int_X \rho_1 \log \rho_1 d\mu_X + \frac{Kt(1 - t)}{2} d_2^W(\nu_0, \nu_1)^2, \end{aligned}$$

where ρ_0 and ρ_1 are densities of ν_0 and ν_1 with respect to μ_X respectively. In particular, for two Borel subsets $A, B \subseteq X$ with $\mu_X(A), \mu_X(B) > 0$, we have

$$(3.2) \quad \begin{aligned} & \log \mu_X(\text{Supp } \nu_t) \\ & \geq (1-t) \log \mu_X(A) + t \log \mu_X(B) + \frac{Kt(1-t)}{2} d_2^W \left(\frac{\mu_X|_A}{\mu_X(A)}, \frac{\mu_X|_B}{\mu_X(B)} \right)^2 \end{aligned}$$

([Stu06b]).

Theorem 1.1 follows from the following key theorem together with Chung-Grigor'yan-Yau's theorem.

Theorem 3.4. *Let (X, μ) be an weighted finite-dimensional Alexandrov space satisfying $\text{CD}(0, \infty)$ and $k \geq 2$. If (X, μ) satisfies*

$$(3.3) \quad \text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_{k+1 \text{ times}}) \leq \frac{1}{D} \log \frac{1}{\kappa}$$

for any $\kappa > 0$, then we have

$$(3.4) \quad \text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_k) \leq \frac{c}{D} \log \frac{1}{\kappa}$$

for any $\kappa > 0$ and for some universal numeric constant $c > 0$.

Proof of Theorem 1.1. In the assumption (3.3) of Theorem 3.4 we can take D as some universal constant times $\sqrt{\lambda_k(X, \mu)}$ by Chung-Grigor'yan-Yau's theorem. Iterating Theorem 3.4 $k-l$ times we get

$$\text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_l) \leq \frac{c^{k-l+1}}{\sqrt{\lambda_k(X, \mu)}} \log \frac{1}{\kappa}$$

for any $\kappa > 0$ and for some universal constant $c > 0$. Since

$$\text{Sep}((X, \mu); \kappa_0, \kappa_1, \dots, \kappa_l) \leq \text{Sep}((X, \mu); \underbrace{\min_i \kappa_i, \min_i \kappa_i, \dots, \min_i \kappa_i}_l)$$

for any $\kappa_0, \kappa_1, \dots, \kappa_l > 0$, we thereby obtain

$$\text{Sep}((X, \mu); \kappa_0, \kappa_1, \dots, \kappa_l) \leq \frac{c^{k-l+1}}{\sqrt{\lambda_k(X, \mu)}} \max_i \log \frac{1}{\kappa_i} \leq \frac{c^{k-l+1}}{\sqrt{\lambda_k(X, \mu)}} \max_{i \neq j} \log \frac{1}{\kappa_i \kappa_j}.$$

This completes the proof of Theorem 1.1. \square

The rough idea of the proof of Theorem 3.4 in [Fun13] for smooth manifolds was the following. It turns out that it is enough to prove (3.4) for sufficiently small $\kappa > 0$ and sufficiently large $c > 0$. We suppose the converse of this, i.e.,

$$\text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_k) > \frac{c}{D} \log \frac{1}{\kappa}$$

for sufficiently small $\kappa > 0$ and sufficiently large $c > 0$. Put $\alpha := (c/D) \log(1/\kappa)$. By the definition of the separation distance there exists k Borel subsets $A_0, A_1, \dots, A_{k-1} \subseteq M$ such that $\min_{i \neq j} d(A_i, A_j) > \alpha$ and $\mu(A_i) \geq \kappa$ for any i . If we choose c is greater than 400, then by the assumption (3.3) we have

$$\text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_{k \text{ times}}, \kappa^{100}) \leq \text{Sep}((X, \mu); \underbrace{\kappa^{100}, \kappa^{100}, \dots, \kappa^{100}}_{k+1 \text{ times}}) \leq \frac{100}{D} \log \frac{1}{\kappa} \leq \frac{\alpha}{4}.$$

Lemma 2.3 implies

$$\mu\left(\bigcup_{i=0}^{k-1} C_{\alpha/4}(A_i)\right) \geq 1 - \kappa^{100}.$$

It means that if $\kappa > 0$ is sufficiently small, the measure of the set $\bigcup_{i=0}^{k-1} C_{\alpha/4}(A_i)$ is nearly 1. Although it is not true, we assume that

$$(3.5) \quad \mu\left(\bigcup_{i=0}^{k-1} C_{\alpha/4}(A_i)\right) = 1$$

in order to tell the idea of the proof. Putting $A := C_{\alpha/4}(A_0)$ and $B := \bigcup_{i=1}^{k-1} C_{\alpha/4}(A_i)$, we have $X = A \cup B$, $A \cap B = \emptyset$, $\mu(A) \geq \kappa$, $\mu(B) \geq \kappa$, and $d(A, B) \geq \alpha/2$.

Let $(\mu_t)_{t \in [0,1]}$ be a geodesic from $\mu_A := (1/\mu(A))\mu|_A$ to μ with respect to d_2^W . For sufficiently small $t > 0$ we have $d(x, A) < \alpha/2 \leq d(A, B)$ for any $x \in \text{Supp } \mu_t$, which gives $\text{Supp } \mu_t \subseteq A$. This leads a contradiction since by (3.2) we have

$$(3.6) \quad \log \mu(A) \geq \log \mu(\text{Supp } \mu_t) \geq (1-t) \log \mu(A) + t \log \mu(X),$$

which implies $\log \mu(A) \geq 0$. Since (3.5) is always not true, we have an error term depending only on κ in (3.6) and we need to consider the trade-off between the error term and t to accomplish the above idea. This leads us to control separated subsets and estimate transport distances between them. In [Fun13], in order to control separated subsets the author heavily relied on E. Milman's theorem in [Mil11](see [Fun13, Claim 3.5]). His theorem is not known for singular metric spaces such as Alexandrov spaces. The key ingredient of his theorem relies on the regularity theory of isoperimetric minimizer. Below we will avoid using his theorem. From A_i we will construct two subsets A, B such that the transport distance between them is at most $c d(A, B)$. The union of A, B does not necessarily have almost total measure.

Proof of Theorem 3.4. It suffices to prove that there exist two universal numeric constants $c_0, \kappa_0 > 0$ such that

$$(3.7) \quad \text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_{k \text{ times}}) \leq \frac{c_0}{D} \log \frac{1}{\kappa}$$

for any $\kappa \leq \kappa_0$. In fact, if $\kappa \geq 1/2$, then the left-hand side of the above inequality is zero and there is nothing to prove. In the case where $\kappa_0 < \kappa \leq 1/2$, by (3.7) we have

$$\begin{aligned} \text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_{k \text{ times}}) &\leq \text{Sep}((X, \mu); \underbrace{\kappa_0, \kappa_0, \dots, \kappa_0}_{k \text{ times}}) \\ &\leq \frac{c_0 \log \frac{1}{\kappa_0}}{D \log \frac{1}{\kappa}} \log \frac{1}{\kappa} \\ &\leq \frac{c_0 \log \frac{1}{\kappa_0}}{D \log 2} \log \frac{1}{\kappa}, \end{aligned}$$

which implies the conclusion of the theorem.

Suppose the contrary to (3.7), i.e.,

$$(3.8) \quad \text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_{k \text{ times}}) > \frac{c_1}{D} \log \frac{1}{\kappa},$$

where $c_1 > 0$ is a sufficiently large universal numeric constant and $\kappa > 0$ is a sufficiently small number. Both the largeness of c_1 and the smallness of κ will be specified later. Note that the assumption (3.8) immediately gives $k\kappa < 1$ (otherwise, the left-hand side of (3.8) is zero). We denote the right-hand side of (3.8) by α , i.e.,

$$\alpha := \frac{c_1}{D} \log \frac{1}{\kappa}.$$

The assumption (3.8) implies the existence of k Borel subsets $A_0, A_1, \dots, A_{k-1} \subseteq X$ such that $\mu(A_i) \geq \kappa$ for any i and $d(A_i, A_j) > \alpha$ for any $i \neq j$. If c_1 is large enough, then after applying Lemma 2.3 to the condition (3.3) we may assume that those A_0, A_1, \dots, A_{k-1} are compact subsets and satisfy $\mu(A_i) \geq \kappa$, $d(A_i, A_j) \geq \alpha/2$, and $\mu(\bigcup_{i=0}^{k-1} A_i) \geq 1 - \kappa^8$. In fact, the assumption (3.3) yields

$$\text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_{k \text{ times}}, \kappa^9) \leq \text{Sep}((X, \mu); \underbrace{\kappa^9, \kappa^9, \dots, \kappa^9}_{k+1 \text{ times}}) \leq \frac{9}{D} \log \frac{1}{\kappa}.$$

Thus whenever $c_1 > 36$ we get

$$\text{Sep}((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_{k \text{ times}}, \kappa^9) < \alpha/4,$$

which implies $\mu(\bigcup_{i=0}^{k-1} C_{\alpha/4}(A_i)) \geq 1 - \kappa^9$ by Lemma 2.3. Note that $\mu(C_{\alpha/4}(A_i)) > \mu(A_i) \geq \kappa$ since μ has full support on X . We can approximate $C_{\alpha/4}(A_i)$ by compact subsets $K_i \subseteq C_{\alpha/4}(A_i)$ so that $\mu(K_i)$ is as close as possible to $\mu(C_{\alpha/4}(A_i))$ ([Bil99, Theorems 1.1 and 1.3]). After taking a sufficient approximation K_i we rechoose A_i as K_i .

For each i we set $\mu_{A_i} := (1/\mu(A_i))\mu|_{A_i}$. Given any i, j we take a 1-Lipschitz function $f_{ij} : X \rightarrow \mathbb{R}$ such that

$$|lm(f_{ij}; \mu_{A_i}) - lm(f_{ij}; \mu_{A_j})| = \sup |lm(f; \mu_{A_i}) - lm(f; \mu_{A_j})|,$$

where the supremum runs over all 1-Lipschitz functions $f : X \rightarrow \mathbb{R}$. We can take such f_{ij} so that $f_{ij} = f_{ji}$.

Put $a_i = a_i(\kappa) := \text{LeRad}((A_i, \mu_{A_i}); -\kappa^8)$.

Claim 3.5. *We have $a_i \leq \alpha/2$ provided that c_1 is large enough.*

Proof. By Lemma 2.4 we have $a_i \leq \text{Sep}(\mu_{A_i}; \kappa^8/2, \kappa^8/2)$. We shall estimate the right-hand side. For any $B, C \subseteq A_i$ such that $\mu_{A_i}(B), \mu_{A_i}(C) \geq \kappa^8/2$ we have $\mu(B), \mu(C) \geq \mu(A_i)\kappa^8/2 \geq \kappa^9/2$. If c_1 is large enough, then by the assumption (3.3) we get

$$\text{Sep}\left((X, \mu); \underbrace{\kappa, \kappa, \dots, \kappa}_{k-1 \text{ times}}, \frac{\kappa^9}{2}, \frac{\kappa^9}{2}\right) \leq \frac{1}{D} \log \frac{2}{\kappa^9} < \frac{\alpha}{2}.$$

Since $d(A_j, A_k) \geq \alpha/2$ we hence obtain $d(B, C) < \alpha/2$, which implies the claim. \square

Setting

$$A'_i := \bigcap_{j=0}^{k-1} \{x \in A_i \mid |f_{ij}(x) - \text{lm}(f_{ij}; \mu_{A_i})| \leq a_i\},$$

we have $\mu_{A_i}(A'_i) \geq 1 - \kappa^8 k$ by the definition of Lévy radius. Recalling that $\kappa < 1/k$ we get $\mu_{A_i}(A'_i) \geq 1 - \kappa^7$. We also get $\mu(\bigcup_{i=0}^{k-1} A'_i) = \sum_{i=0}^{k-1} \mu(A_i) \mu_{A_i}(A'_i) \geq 1 - \kappa^8 - \kappa^7 \geq 1 - \kappa^6$ provided that κ is small enough.

Take $x_i \in A'_i$ and $x_j \in A'_j$ such that $d(x_i, x_j) = d(A'_i, A'_j)$. Claim 3.5 yields that for any 1-Lipschitz functions $f : X \rightarrow \mathbb{R}$ we have

$$\begin{aligned} (3.9) \quad |\text{lm}(f; \mu_{A_i}) - \text{lm}(f; \mu_{A_j})| &\leq |\text{lm}(f_{ij}; \mu_{A_i}) - \text{lm}(f_{ij}; \mu_{A_j})| \\ &\leq a_i + a_j + |f_{ij}(x_i) - f_{ij}(x_j)| \\ &\leq a_i + a_j + d(A'_i, A'_j) \\ &\leq 3 d(A'_i, A'_j). \end{aligned}$$

Without loss of generality one may assume that $d(A'_0, A'_1) = \min_{i \neq j} d(A'_i, A'_j)$ and $\mu(A'_0) \leq \mu(A'_1)$. Set $B_0 := A'_0$ and $B_1 := A'_0 \cup A'_1$.

Claim 3.6. *There exists a coupling π of μ_{B_0} and μ_{B_1} such that*

$$\pi(\{(x, y) \in X \times X \mid d(x, y) \leq 8 d(A'_0, A'_1)\}) \geq 1 - \kappa^6.$$

Proof. Put $\delta := 8 d(A'_0, A'_1)$. It suffices to prove that $\text{di}_{\kappa^6/\delta}(\mu_{B_0}, \mu_{B_1}) \leq \delta$ according to Theorem 2.7. In fact, Theorem 2.7 gives that there exists a δ -transportation π_0 from μ_{B_0} to μ_{B_1} such that $\text{def } \pi_0 \leq \kappa^6$. If $\text{def } \pi_0 = 0$, then we set $\pi := \pi_0$. If $\text{def } \pi_0 > 0$, then set

$$\pi := \pi_0 + \frac{1}{\text{def } \pi_0} (\mu_{B_0} - (\text{proj}_1)_* \pi_0) \times (\mu_{B_1} - (\text{proj}_2)_* \pi_0).$$

It is easy to check that π fulfills the desired property.

Given a Borel subset $A \subseteq B_1$ we shall prove that

$$(3.10) \quad \mu_{B_1}(C_\delta(A)) \geq \mu_{B_0}(A) - \kappa^6.$$

As we remarked just after Definition 2.5, this implies the other inequality $\mu_{B_0}(C_\delta(A)) \geq \mu_{B_1}(A) - \kappa^6$ and hence $\text{di}_{\kappa^6/\delta}(\mu_{B_0}, \mu_{B_1}) \leq \delta$.

To prove (3.10) we may assume that $\mu_{B_0}(A) \geq \kappa^6$. Define $f : X \rightarrow \mathbb{R}$ by $f(x) := d(x, C_{\delta/2}(A) \cap A'_0)$. Note that

$$(3.11) \quad \mu_{A_0}(A) \geq (\mu(B_0)/\mu(A_0))\mu_{B_0}(A) \geq (1 - \kappa^7)\kappa^6.$$

As we showed in the proof of Claim 3.5 we get

$$\text{Sep}(\mu_{A_0}; \kappa^6(1 - \kappa^7), \kappa^7) \leq \text{Sep}(\mu_{A_0}; \kappa^7, \kappa^7) < \alpha < \delta/2,$$

which gives $\mu_{A_0}(C_{\delta/2}(A)) \geq 1 - \kappa^7$ by (3.11). Since $\mu_{A_0}(C_{\delta/2}(A) \cap A'_0) \geq \mu_{A_0}(C_{\delta/2}(A)) - \kappa^7 \geq 1 - 2\kappa^7 > 1/2$ for sufficiently small κ we have $lm(f; \mu_{A_0}) = 0$. Using (3.9) we then obtain

$$lm(f; \mu_{A_1}) \leq 3 d(A'_0, A'_1).$$

Put $B := \{x \in X \mid d(x, C_{\delta/2}(A) \cap A'_0) \leq \delta/2\}$. Then

$$\begin{aligned} \mu_{A_1}(B) &\geq \mu_{A_1}(\{x \in X \mid f(x) \leq 3 d(A'_0, A'_1) + a_1\}) \\ &\geq \mu_{A_1}(\{x \in X \mid |f(x) - lm(f; \mu_{A_1})| \leq a_1\}) \geq 1 - \kappa^8, \end{aligned}$$

which shows

$$\mu(B \cap A'_1) \geq \mu(B \cap A_1) - \mu(A_1 \setminus A'_1) \geq (1 - \kappa^8)\mu(A_1) - \kappa^7\mu(A_1) \geq (1 - 2\kappa^7)\mu(A_1).$$

Combining this inequality with $\mu(B \cap A'_0) \geq \mu(C_\delta(A) \cap A'_0) \geq (1 - 2\kappa^7)\mu(A_0)$ we obtain

$$\mu_{B_1}(B) \geq \mu(B_1)^{-1} \{\mu(A'_0)(1 - 2\kappa^7) + \mu(A'_1)(1 - 2\kappa^7)\} \geq 1 - 4\kappa^7 \geq \mu_{B_0}(A) - \kappa^6$$

provided that κ is sufficiently small. This completes the proof of the claim. \square

We set $\Delta := \{(x, y) \in X \times X \mid d(x, y) \leq 8 d(A'_0, A'_1)\}$. We consider two Borel probability measures $\mu_0 := a(\text{proj}_1)_*(\pi|_\Delta)$ and $\mu_1 := a(\text{proj}_2)_*(\pi|_\Delta)$, where $a := \pi(\Delta)^{-1}$. By Claim 3.6 we have

$$(3.12) \quad 1 \leq a \leq 1/(1 - \kappa^6)$$

and

$$(3.13) \quad d_2^W(\mu_0, \mu_1)^2 \leq a \int_{X \times X} d(x, y)^2 d\pi|_\Delta(x, y) \leq 8^2 d(B_0, B_1 \setminus B_0)^2.$$

Take an optimal dynamical transference plan Π such that $(e_i)_*\Pi = \mu_i$ for each $i = 0, 1$. Putting $r := d(B_0, B_1 \setminus B_0)$, we consider the set

$$\Gamma_t := \{\gamma \in \text{Supp } \Pi \mid d(e_0(\gamma), e_t(\gamma)) \leq r/2\}.$$

By (3.13) we have

$$(r^2/4)\Pi(\Gamma \setminus \Gamma_t) \leq d_2^W((e_0)_*\Pi, (e_t)_*\Pi)^2 = t^2 d_2^W(\mu_0, \mu_1)^2 \leq \{8t d(B_0, B_1 \setminus B_0)\}^2,$$

which yields

$$(3.14) \quad \Pi(\Gamma_t) \geq 1 - ct^2,$$

where $c := 8^3$. For $s \in [0, 1]$ we put $\nu_s := (e_s)_* \frac{\Pi|_{\Gamma_t}}{\Pi(\Gamma_t)}$. By the definition of ν_s we obtain the following.

Claim 3.7. $\text{Supp } \nu_t \cap \bigcup_{i=0}^{k-1} A'_i \subseteq B_0$.

By using Claim 3.7, we get

$$\begin{aligned}
 (3.15) \quad \log \mu(B_0) + \frac{\kappa^6}{\mu(B_0)} &\geq \log \mu(B_0) + \log \left(1 + \frac{\kappa^6}{\mu(B_0)}\right) \\
 &= \log(\mu(B_0) + \kappa^6) \\
 &\geq \log \left\{ \mu \left(\text{Supp } \nu_t \cap \bigcup_{i=0}^{k-1} A'_i \right) + \mu \left(\text{Supp } \nu_t \setminus \bigcup_{i=0}^{k-1} A'_i \right) \right\} \\
 &= \log \mu(\text{Supp } \nu_t)
 \end{aligned}$$

Note that $(\nu_s)_{s \in [0,1]}$ is a geodesic between ν_0 and ν_1 . Since

$$(3.16) \quad \nu_i = \frac{(e_i)_* \Pi|_{\Gamma_t}}{\Pi(\Gamma_t)} \leq \frac{(e_i)_* \Pi}{\Pi(\Gamma_t)} = \frac{\mu_i}{\Pi(\Gamma_t)} \leq \frac{a}{\Pi(\Gamma_t)} (\text{proj}_{i+1})_* \pi = \frac{a}{\Pi(\Gamma_t)} \mu_{B_i}$$

for $i = 0, 1$, each ν_i is absolutely continuous with respect to μ , and especially the above geodesic $(\nu_s)_{s \in [0,1]}$ is unique. For each $i = 0, 1$, we write $d\nu_i = \rho_i d\mu$. By (3.1), we get

$$(3.17) \quad \log \mu(\text{Supp } \nu_t) \geq -(1-t) \int_X \rho_0 \log \rho_0 d\mu - t \int_X \rho_1 \log \rho_1 d\mu.$$

For a subset $A \subseteq X$ we denote by 1_A the characteristic function of A , i.e., $1_A(x) := 1$ if $x \in A$ and $1_A(x) := 0$ if $x \in X \setminus A$.

Claim 3.8. *We have*

$$\rho_i \log \rho_i \leq \frac{c_t 1_{B_i}}{\mu(B_i)} \log \frac{c_t 1_{B_i}}{\mu(B_i)} \quad (i = 0, 1),$$

where $c_t := a/\Pi(\Gamma_t)$.

Proof. By (3.16) we have $\rho_i \leq (c_t/\mu(B_i))1_{B_i}$. Since $c_t \geq 1$ and $u \log u \leq v \log v$ for any two positive numbers u, v such that $u \leq v$ and $v \geq 1$, we obtain the claim. \square

Combining Claim 3.8 with (3.15) and (3.17) we have

$$\begin{aligned}
 \log \mu(B_0) + \frac{\kappa^6}{\mu(B_0)} &\geq -(1-t) \int_X \frac{c_t 1_{B_0}}{\mu(B_0)} \log \frac{c_t 1_{B_0}}{\mu(B_0)} d\mu - t \int_X \frac{c_t 1_{B_1}}{\mu(B_1)} \log \frac{c_t 1_{B_1}}{\mu(B_1)} d\mu \\
 &= -c_t \log c_t + c_t(1-t) \log \mu(B_0) + c_t t \log \mu(B_1).
 \end{aligned}$$

Substituting $t := \kappa^3$, we thereby obtain

$$\begin{aligned}
 (3.18) \quad \log(1/2) + 4\kappa^2 &\geq \log \frac{\mu(B_0)}{\mu(B_1)} + \frac{\kappa^6}{\kappa^3 \mu(B_0)} \\
 &\geq -\frac{c_t}{\kappa^3} \log c_t + \frac{c_t - 1}{\kappa^3} (1 - \kappa^3) \log \mu(B_0) + (c_t - 1) \log \mu(B_1).
 \end{aligned}$$

Using (3.12) and (3.14) we estimate each term on the right-side of the above inequalities as

$$\frac{c_t \log c_t}{\kappa^3} = \frac{1}{(1 - \kappa^6)(1 - c\kappa^6)} \cdot \frac{-\log(1 - \kappa^6)(1 - c\kappa^6)}{\kappa^3} \leq \frac{2\kappa^3 + 2c\kappa^3}{(1 - \kappa^6)(1 - c\kappa^6)}$$

$$\left| \frac{c_t - 1}{\kappa^3} \log \mu(B_0) \right| \leq \frac{a - \Pi(\Gamma_t)}{\kappa^3 \Pi(\Gamma_t)} \log \frac{2}{\kappa} \leq \frac{\kappa^3 (\{1 + c(1 - \kappa^6)\})}{(1 - \kappa^6)(1 - c\kappa^6)} \log \frac{2}{\kappa}$$

and

$$|(c_t - 1) \log \mu(B_1)| \leq \frac{\kappa^6 \{1 + c(1 - \kappa^6)\}}{(1 - \kappa^6)(1 - c\kappa^6)} \log \frac{1}{\kappa}.$$

These estimates imply the right-side of the inequalities (3.18) is close to zero for sufficiently small $\kappa > 0$. Since the left-side of the inequality (3.18) is about $\log(1/2) < 0$ for sufficiently small $\kappa > 0$, this is a contradiction. This completes the proof of the theorem. \square

4. CONJECTURE

We raise the following conjecture for eigenvalues of Laplacian.

Conjecture 4.1. *If (X, μ) is an weighted compact finite-dimensional Alexandrov space of $\text{CD}(0, \infty)$ and k is a natural number, then we have*

$$\lambda_{k+1}(X, \mu) \leq c \lambda_k(X, \mu)$$

for some universal constant $c > 0$.

The answer is positive for any compact Riemannian homogeneous manifolds ([CY05], [Li80]).

To explain how Theorem 1.1 relates with the above conjecture we need to explain some basics on the theory of concentration of measure in the sense of Lévy and V. Milman ([Lev51], [Mil71]). Refer to [Led01] for details.

We denote the open r -neighborhood of a subset A in a metric space by $O_r(A)$.

Definition 4.2 (Concentration function, [AM80]). Let X be an mm-space. For $r > 0$ we define the real number $\alpha_X(r)$ as the supremum of $\mu_X(X \setminus O_r(A))$, where A runs over all Borel subsets of X such that $\mu_X(A) \geq 1/2$. The function $\alpha_X : (0, +\infty) \rightarrow \mathbb{R}$ is called the *concentration function*.

The following lemma asserts that exponential concentration inequalities and logarithmic 1-separation inequalities are equivalent:

Lemma 4.3. *Let X be an mm-space.*

(1) *If X satisfies*

$$(4.1) \quad \text{Sep}(X; \kappa, \kappa) \leq \frac{1}{C} \log \frac{c}{\kappa}$$

for any $\kappa > 0$, then we have $\alpha_X(r) \leq c \exp(-Cr)$ for any $r > 0$.

(2) Conversely, if X satisfies $\alpha_X(r) \leq c' \exp(-C'r)$ for any $r > 0$, then we have

$$\text{Sep}(X; \kappa, \kappa) \leq \frac{2}{C'} \log \frac{c'}{\kappa}$$

for any $\kappa > 0$.

Proof. (1) Assume that X satisfies (4.1) and let $A \subseteq X$ be a Borel subset such that $\mu_X(A) \geq 1/2$. For $r > 0$ we put $\kappa := \mu_X(X \setminus O_r(A))$. Since

$$r \leq d(X \setminus O_r(A), A) \leq \text{Sep}(X; \kappa, 1/2) \leq \text{Sep}(X; \kappa, \kappa) \leq \frac{1}{C} \log \frac{c}{\kappa},$$

we have $\kappa \leq c \exp(-Cr)$, which gives the conclusion of (1).

(2) Assuming that $\alpha_X(r) \leq c' \exp(-C'r)$, we take two Borel subsets $A, B \subseteq X$ such that $\mu_X(A) \geq \kappa$, $\mu_X(B) \geq \kappa$, and $d(A, B) = \text{Sep}(X; \kappa, \kappa)$. Let \tilde{r} be any positive number satisfying

$$\alpha_X(\tilde{r}) \leq c' \exp(-C'\tilde{r}) < \kappa,$$

i.e.,

$$\tilde{r} > \frac{1}{C'} \log \frac{c'}{\kappa}.$$

Since $\mu_X(A) \geq \kappa$, by [Led01, Lemma 1.1], we have

$$1 - \mu_X(O_{2\tilde{r}}(A)) \leq \alpha_X(\tilde{r}) < \kappa.$$

Hence we have

$$\mu_X(O_{2\tilde{r}}(A) \cap B) > (1 - \kappa) + \kappa - 1 = 0,$$

which yields $\text{Sep}(X; \kappa, \kappa) = d(A, B) \leq 2\tilde{r}$. Letting $\tilde{r} \rightarrow C'^{-1} \log(c'/\kappa)$ we obtain (2). \square

In the series of works [Mil10, Mil11, Mil12a], E. Milman proved that a uniform tail-decay of the concentration function implies the linear isoperimetric inequality (Cheeger's isoperimetric inequality) under assuming the non-negativity of Bakry-Émery Ricci curvature. Note that the linear isoperimetric inequality always implies an appropriate Poincaré inequality and thus a lower bound for the first eigenvalue of the weighted Laplacian. The key ingredient of E. Milman's approach to the above result is the concavity of isoperimetric profile under the assumption of the non-negativity of Bakry-Émery Ricci curvature, the fact based on the regularity theory of isoperimetric minimizers (see [Mil10, Appendix]). See also [Led01] for the heat semigroup approach. In [GRS11] Gozlan, Roberto, and Samson proved that any exponential concentration inequalities imply appropriate Poincaré inequalities under assuming $\text{CD}(0, \infty)$. In other words if an mm-space X satisfying $\text{CD}(0, \infty)$ enjoys a logarithmic 1-separation inequality $\text{Sep}(X; \kappa, \kappa) \leq (1/D) \log(1/\kappa)$ then we have $\sqrt{\lambda_1(X)} \geq cD$, where c is some universal constant and $\lambda_1(X)$ is the spectral gap. Especially combining Theorem 1.1 for $k = 2$ and $l = 1$ with this theorem yields the positive answer to Conjecture 4.1 for $k = 1$. In general according to Theorem 1.1 in order to give an affirmative answer to Conjecture 4.1 it suffices to extend E. Milman's theorem or more weakly Gozlan-Roberto-Samson's theorem in terms of $\lambda_k(X, \mu)$, i.e., any

logarithmic k -separation inequalities imply appropriate estimates of the k -th eigenvalue $\lambda_k(X, \mu)$ from below under assuming $CD(0, \infty)$.

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